MODEL QUESTION

B.Sc. Sem-VI Paper – 16 (XVI)

Group – B (Metric Space)

1) Let 'd' be a metric on a non-empty set E and the function f defined be on $E \times E$ by

$$f(x,y) = \min\{1, d(x,y)\} \quad \forall\ x,y \in E$$

Then prove that f is also a metric on E.

2) Let (X,f) be a metric space then Prove that (X,d) be a metric space where d is defined by

$$d(x,y) = \frac{f(x,y)}{1+f(x,y)} \quad \forall\ x,y \in X.$$

3) Let $(u_1, u_2, u_3, \cdots u_n)$ and $(v_1, v_2, v_3, \cdots v_n)$ be any two n-tuples of real numbers then prove that

$$\sum_{i=1}^{n} (u_i v_i) \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

or,

$$\sum_{i=1}^{n} u_i v_i \leq \left(\sqrt{\sum_{i=1}^{n} u_i^2}\right)\left(\sqrt{\sum_{i=1}^{n} v_i^2}\right)$$

4) Let R^n be the set of all n-tuples of real numbers given by $x = (x_1, x_2, x_3, \cdots x_n)\ \forall\ x_i \in R$, $i = 1,2,3 \cdots n$

we define a mapping d_2 from $R^n \times R^n$ to R. ie. $d_2 : R^n \times R^n \longrightarrow R$

given by

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Then show that (R^n, d_2) is a metric space.

5) i) Define an open sphere and open set. Show that every open sphere in a metric space (X,d) is an open set.

ii) Prove that A subset G of a metric space (X,d) is open iff G is a union of open spheres.

6) Define Neighbourhood of a point. Let (X,d) be a metric space then Prove that for each pair x,y of distinct points of X, $\exists$ a nhd U of x and a nhd V of y such that ~~UA~~ $U \cap V = \phi$.

7) Let (X,d) be a metric space and G be any non-empty subset of X then P.T. G is open iff each point of G posses a nhd which is contained in G.

8) Define limit point of a set.
Let (X,d) be any metric space and F is any subset of X then F is closed iff it contains each of its limit point. Prove it.

9) Show that the set of all limit points of a set is a closed set.
i.e. the derived set is a closed set.

10) Let (X,d) be a metric space and F be any subset of X then Prove that F is closed iff $F = \bar{F}$.

11) Let (X,d) be a metric space and let $\langle x_n \rangle$ be a sequence of points of X. If the sequence $\langle x_n \rangle$ is convergent then Prove that it converges to unique limit

12) Define Cauchy sequence and a complete metric space.
Show that in a metric space (X,d) every convergent sequence is Cauchy Sequence. Also show that the converse is not always true.

13) State and Prove Cantor's Intersection Theorem.

14) Let (X,d_1) and (Y,d_2) be any two metric spaces. A function ~~d~~ $f: X \longrightarrow Y$ is Continuous iff $f^{-1}(O)$ is an open subset of X whenever O is an open subset of Y. Prove it

15). Let (X,d_1) and (Y,d_2) be any two metric spaces and let $f: X \rightarrow Y$ be any mapping from X into Y then Prove that the mapping f is Continuous at $c \in X$ iff every sequence $\langle x_n \rangle$ of points of X, we have

$$x_n \longrightarrow c \Rightarrow f(x_n) \rightarrow f(c).$$